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On Unification: Equational Theories Are Not Bounded†

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We are interested in first-order unification problems and, more specifically, in the hierarchy of equational theories based on the cardinality of the set of most general unifiers. The following result is established in this paper: if T is a suitable first-order equational theory that is not unitary, then T is not bounded; that is, there is no integer $n > 1$ such that for every unification problem $\langle s = t \rangle_T$, the cardinality of the set of most general unifiers for $\langle s = t \rangle_T$ is at most n. Hence, the class of (non-unitary) finitary theories cannot be decomposed into a hierarchy obtained by uniformly bounding the cardinalities of the sets of most general unifiers.

1. Unification in Equational Theories

1.1. Motivation

Unification theory is concerned with problems of the following type. Let f and g be function symbols, a and b be constants, and x and y be variables. Consider two first-order terms built from these symbols, for example

$$t_1 = f(x, g(a, b))$$
$$t_2 = f(g(y, b), x).$$

The problem is to decide whether there exist terms which can be substituted for the variables x and y such that the two terms thus obtained from t_1 and t_2 become equal. In the example, $g(a, b)$ and a are two such terms. We shall write

$$\sigma_1 = \{x \leftarrow g(a, b), y \leftarrow a\}$$

for such a unifying substitution: σ_1 is a *unifier* of t_1 and t_2 since $\sigma_1 t_1 = \sigma_1 t_2$.

In addition to the above decision problem, there is also the problem of finding a unification algorithm which enumerates the unifiers for a given pair t_1 and t_2.

Consider a variation of the above example which arises when we assume that f is commutative

(C) $$f(x, y) = f(y, x).$$

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Now σ_1 is still a unifying substitution, but in addition $\sigma_2 = \{y \leftarrow a\}$ is also a unifier for t_1 and t_2 since

$$\sigma_2 t_1 = f(x, g(a, b)) =_C f(g(a, b), x) = \sigma_2 t_2.$$

But σ_2 is more general than σ_1 since σ_1 is an instance of σ_2 obtained as the composition $\lambda \circ \sigma_2$ with $\lambda = \{x \leftarrow g(a, b)\}$; hence, a unification algorithm for t_1 and t_2 only needs to compute σ_2.

In many cases there is a single and essentially unique least upper bound on the generality lattice of unifiers, called *the most general unifier*. Under commutativity, however, there are pairs of terms which have more than one most general unifier, but they always have at most finitely many. This is in contrast, for example, to the above situation of free terms where every pair has at most one most general unifying substitution.

The problem becomes entirely different when we assume that the function denoted by f is associative

$$\text{(A)} \qquad f(x, f(y, z)) = f(f(x, y), z).$$

In this case, σ_1 is still a unifying substitution, but

$$\sigma_3 = \{x \leftarrow f(g(a, b)), g(a, b)), y \leftarrow a\}$$

is also a unifier since

$$\sigma_3 t_1 = f(f(g(a, b), g(a, b)), g(a, b)) =_A f(g(a, b), f(g(a, b), g(a, b))) = \sigma_3 t_2.$$

But

$$\sigma_4 = \{x \leftarrow f(g(a, b), f(g(a, b), g(a, b))), y \leftarrow a\}$$

is also a unifier; in fact, there are infinitely many unifiers, all of which are most general.

If we assume that both axioms (A) and (C) hold for f, then the situation changes once again and for any pair of terms there are at most finitely many most general unifiers under (A) and (C).

In this paper we establish the following result: if T is a first-order theory such that the set of most general unifiers has more than one element, then T is not bounded. That is, there is no integer $n > 1$ such that for every unification problem $\langle s = t \rangle_T$ the set of most general unifiers is at most n.

THEOREM. *If T is a suitable first-order equation theory that is not unitary, then T is not bounded.*

Our interest in this result stems from the description of the unification hierarchy (Siekmann, 1984), where it is argued that one of the major open problems of unification theory is to characterise the border between finitary and infinitary theories as well as between unitary and finitary theories. We show that the class of (non-unitary) finitary theories cannot be decomposed into a hierarchy obtained by uniformly bounding the cardinalities of the sets of most general unifiers. Hence, one cannot use the notion of "bounded size" to characterise the difference between finitary and unitary theories.

1.2. DEFINITIONS AND NOTATION

Unification theory rests upon the usual algebraic notation (see, e.g., Grätzer, 1968) with the familiar concept of an algebra $\mathcal{A} = (A, \Omega)$, where A is the *carrier* and Ω is a family of *operators* given with their arities.

As usual, let F_Ω denote the algebra with carrier the terms (built up from the set V of variables and the symbols in Ω) and with operators the term constructors corresponding to each operator of Ω. This algebra is the *absolutely free* (*term*) *algebra* since it gives an algebraic structure to the terms. If the carrier is ground, i.e. there are no variables in the terms, then it is called the initial algebra or Herbrand universe.

Let $\hat{\sigma}: V \to F_\Omega$ be a mapping equal to the identity almost everywhere. A *substitution* $\sigma: F_\Omega \to F_Q$ is the endomorphic extension of $\hat{\sigma}$ to F_Ω and is represented as a set of pairs: $\sigma = \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$; Σ is the *set of substitutions* on F_Ω and ε the identity.

An *equation* $s = t$ is a pair of terms. For a set of equations T, the *equational theory presented by* T (in short, the equational theory T) is defined as the finest congruence on F_Ω containing all pairs $\sigma s = \sigma t$ for $s = t$ in T and σ in Σ; this congruence is denoted by $=_T$. The quotient algebra of F_Ω by the congruence $=_T$ is denoted $F_\Omega/=_T$. An equation $s = t$ is *valid* in an algebra $\mathscr{A}$ (in symbols, $\mathscr{A} \models s = t$) if for every homomorphism $\rho: F_\Omega \to \mathscr{A}$, $\rho s = \rho t$ in $\mathscr{A}$; we also say that $\mathscr{A}$ is a *model* of $s = t$. Algebra $\mathscr{A}$ is a model of a set of equations E if $\mathscr{A} \models s = t$ for every $s = t$ in E. The class of models of E is called the *variety* defined by E.

An equation $s = t$ is *unifiable* (*solvable*) in an algebra $\mathscr{A}$ if there exists a substitution $\sigma \in \Sigma$ such that $\mathscr{A} \models \sigma s = \sigma t$. For a given set of equations T (that is, a given equational theory T) a *unification problem* for T is denoted as $\langle s = t \rangle_T$, where $s, t \in F_\Omega$; the problem is to decide whether $s = t$ is unifiable in $F_\Omega/=_T$.

A substitution $\sigma \in \Sigma$ is called a *T-unifier* for $\langle s = t \rangle_T$ if and only if $\sigma s =_T \sigma t$ (if and only if $F_\Omega/=_T \models \sigma s = \sigma t$). The subset of Σ which unifies $\langle s = t \rangle$ is denoted by $U\Sigma_T(s, t)$ and is called the *set of unifiers* (for s and t) under T. (We will omit the subscript T and (s, t) if they are clear from the context.) The composition of substitutions is defined by the usual composition of mappings: $(\sigma \circ \tau)t = \sigma(\tau t)$. For a set W of variables, *T-equality* is extended to substitutions by $\sigma =_T \tau[W]$ if and only if $\forall\, x \in W$: $\sigma x =_T \tau x$, in which case we say that σ and τ are *T-equal* in W.

Let $\leqslant_T$ be a partial order on terms such that $s \leqslant_T t$ if and only if there exists $\delta \in \Sigma$ satisfying $s =_T \delta t$. This relation is extended to substitutions: we say that σ is an *instance* of τ and τ is *more general than* σ, in symbols $\sigma \leqslant_T \tau[W]$, if and only if there exists $\lambda \in \Sigma$ such that $\sigma =_T \lambda \circ \tau[W]$ for some $W \subseteq X$. If $\sigma \leqslant_T \tau[W]$ and $\tau \leqslant_T \sigma[W]$, then $\sigma \sim_T \tau[W]$ and σ and τ are called *T-equivalent in* $[W]$.

For a given unification problem $\langle s = t \rangle_T$, we do not want to compute the whole set of unifiers $U\Sigma_T(s, t)$, but rather a smaller set that is useful in representing $U\Sigma$. For this reason we define $CU\Sigma_T(s, t)$, the *complete set of unifiers of* s *and* t for $W = \mathrm{Var}(s, t)$ as follows:

(i) $CU\Sigma \subseteq U\Sigma$; (*correctness*)

(ii) $\forall\, \delta \in U\Sigma$ there exists $\sigma \in CU\Sigma$
such that $\delta \leqslant_T \sigma[W]$ (*completeness*)

The *set of most general unifiers* $\mu U\Sigma_T(s, t)$ is defined by (i), (ii), and the following condition:

(iii) $\forall\, \sigma, \delta \in \mu U\Sigma$, if $\sigma \leqslant_T \delta[W]$,
then $\sigma =_T \delta[W]$. (*minimality*)

For theoretical reasons (e.g. idempotency of substitutions) as well as for many practical applications, it useful to have the additional technical requirement that only new variables are introduced by the unifier σ. Let

$$VCOD(\sigma) = \{v \mid \text{for some } x \in V, v \in \text{Var}(\sigma x)\}.$$

Then consider the following condition:

(iv) for $\sigma \in \mu U\Sigma_T(s, t)$ and a set Z of variables,
$VCOD(\sigma) \cap Z = \phi$. *(protection of Z)*

If conditions (i)–(iv) are fulfilled we say that $\mu U\Sigma$ is a *set of most general unifiers away from* Z (Plotkin, 1972; Huet, 1976).

The set $\mu U\Sigma_T$ does not always exist; if it does, then it is unique up to equivalence $\sim_T$ (see Fages & Huet, 1983). For that reason it is sufficient to generate one set $\mu U\Sigma_T$.

Central to unification theory is the notion of the hierarchy of equational theories based on $\mu U\Sigma$:

(i) a theory T is *unitary* if $\mu U\Sigma$ always exists and has at most one element;
(ii) a theory T is *finitary* if $\mu U\Sigma$ always exists and is finite, and T is not unitary;
(iii) a theory T is *infinitary* if $\mu U\Sigma$ always exists and there exists a pair of terms such that $\mu U\Sigma$ is infinite for this pair;
(iv) a theory T is *type zero* otherwise.

The field of unification theory and its applications are surveyed in Raulefs *et al.* (1979), Siekmann & Szabo (1981), and Siekmann (1984).

1.3. THE PROBLEM

Let $|\mu U\Sigma|$ denote the cardinality of the set $\mu U\Sigma$. We say that a given unification problem $\langle s = t \rangle_T$ is *bounded* if there exists an integer n such that $|\mu U\Sigma_T(s, t)| \leqslant n$. An equational theory T is *bounded* if there exists an integer n such that

$$\forall\, s, t \in F_\Omega |\mu U\Sigma_T(s, t)| \leqslant n.$$

In this paper we are interested in the question of whether the class of finitary equational theories can be subclassified into bounded theories. In other words, can the hierarchy of unification problems described above be decomposed, or, is it the *finest* structure based on the cardinality of $\mu U\Sigma$?

2. Bounded Unification Problems

Consider the following examples of (trivially) bounded problems.

2.1. COMMUTATIVITY

Let $\mathbf{C} = \{f(x, y) = f(y, x)\}$ and consider the unification problem $\langle f(x, y) = f(a, b) \rangle_C$ which is bounded by $n = 2$:

$$\mu U\Sigma = \{\{x \leftarrow a, y \leftarrow b\}, \{x \leftarrow b, y \leftarrow a\}\}.$$

If we take the set of terms as those in which "f" occurs at most once, then it is easy to see that $\mathbf{C}$ is bounded in size by two for *this set of terms*. Let h be a binary function symbol which is free in $\mathbf{C}$, i.e. it does not occur anywhere in $\mathbf{C}$ and define

$$s = f(x, y), \quad t = f(a, b), \quad \hat{s} = f(u, v), \quad \hat{t} = f(a, b).$$

The unification problem $\langle h(\hat{s}, s) = h(\hat{t}, t)\rangle_C$ has $n^2 = 2^2 = 4$ most general unifiers (and so is not bounded by $n = 2$):

$$\mu U\Sigma = \{\{x \leftarrow a, y \leftarrow b, u \leftarrow a, v \leftarrow b\}, \{x \leftarrow a, y \leftarrow b, u \leftarrow b, v \leftarrow a\}, \{x \leftarrow b, y \leftarrow a, u \leftarrow a, v \leftarrow b\}, \{x \leftarrow b, y \leftarrow a, u \leftarrow b, v \leftarrow a\}\}.$$

(This construction is essentially that used in the proof of the main lemma below.)

2.2. ASSOCIATIVITY

Let

$$\mathbf{A} = \{f(f(x, y), z) = f(x, f(y, z))\}$$

and abbreviate $f(a, a)$ as aa and $aa \ldots a$ (n times) as a^n. Then $\langle xa = ax\rangle_A$ is unbounded (Plotkin, 1972), since $\mu U\Sigma = \{x \leftarrow a^n | n \geqslant 1\}$. However, for any fixed value of $n \geqslant 2$, $\langle xa^n = a^n y\rangle_A$ is bounded in size by $n+1$ since

$$\mu U\Sigma = \{\{x \leftarrow a, y \leftarrow a\}, \{x \leftarrow a^2, y \leftarrow a^2\}, \ldots, \{x \leftarrow a^n, y \leftarrow a^n\}, \{x \leftarrow a^n v, y \leftarrow v a^n\}\}.$$

2.3. ASSOCIATIVITY AND COMMUTATIVITY

Let

$$\mathbf{AC} = \{f(f(x, y), z) = f(x, f(y, z)), f(x, y) = f(y, x), f(x, 1) = x\},$$

where 1 is an identity element. Using the same abbreviations as above, the problem $\langle x^2 ya = b^2 z\rangle_{AC}$, where $x, y, z \in V$ and $a, b \in C$, is bounded by two since

$$\mu U\Sigma = \{\{x \leftarrow vb, y \leftarrow u, z \leftarrow uv^2 a\}, \{x \leftarrow v, y \leftarrow ub^2, z \leftarrow uv^2\}\}.$$

In general, the cardinality of $\mu U\Sigma$ is determined by the dimension of the solution space of certain diophantine equations (see Stickel, 1981, and Herold & Siekmann, 1985, for details).

To summarise, notice that the above examples demonstrate that there are bounded unification problems, i.e. a *given* problem may be bounded. The examples also demonstrate that it is possible to give a subset of first-order terms such that an equational theory is bounded *on this set*. However, it is shown in the following section that it is impossible to find an equational non-unitary theory that is bounded on the *whole* set of first-order terms.

3. Equational Theories Are Not Bounded

The main result follows immediately from Lemma 3.1 below, but first we must describe those theories to which the lemma applies. Thus, we refer to a theory as being *suitable* if it is a first-order theory with at least one binary (or larger arity) function symbol that is free and if there are no bounds on the number of times an individual variable or an individual constant or an individual function symbol may occur.

LEMMA 1. *Let T be a suitable theory. For any integer $n \geqslant 1$ and any problem $\langle s = t\rangle_T$ such that $\mu U\Sigma(s, t)$ has cardinality n, there exists a problem $\langle s' = t'\rangle_T$ such that $\mu U\Sigma(s', t')$ has cardinality n^2.*

PROOF. Let $\langle s = t\rangle_T$ be a problem in T such that $\mu U\Sigma(s, t)$ away from $Z = \mathrm{Var}(s, t)$ exists and has cardinality n. If no variables occur in either s or t, then n is equal to 1 so that setting s' identically equal to s and t' identically equal to t yields the result trivially. Thus, we assume that either s or t contains occurrences of variables.

Let $\mathrm{Var}(s, t) = \{x_1, \ldots, x_k\} = Z$ be the variables that occur in s and t. Let U be the union of the variables that are introduced by the unifiers in $\mu U\Sigma(s, t)$, i.e.

$$U = \{VCOD(\sigma) | \sigma \in \mu U\Sigma\}.$$

Let $y_1, \ldots, y_k$ be k new variables, so that

$$\{y_1, \ldots, y_k\} \cap (Z \cup U) = \phi.$$

Define $\hat{s}$ to be the result of substitution y_i for each occurrence of x_i in s, $1 \leqslant i \leqslant k$, and define $\hat{t}$ similarly.

Since $\mu U\Sigma(s, t)$ exists and has cardinality n, it follows immediately that $\mu U\Sigma(\hat{s}, \hat{t})$ away from $Z = \mathrm{Var}(\hat{s}, \hat{t}) \cup U$ exists and has cardinality n. Let h be a free binary function symbol. Consider the problem $\langle h(s, \hat{s}) = h(t, \hat{t})\rangle_T$. Notice that if $\alpha \in \mu U\Sigma(s, t)$ and $\beta \in \mu U\Sigma(\hat{s}, \hat{t})$, then $\alpha \circ \beta$ is in $CU\Sigma(h(s, \hat{s}), h(t, \hat{t}))$. We will show that in fact

$$M = \{\alpha \circ \beta | \alpha \in \mu U\Sigma(s, t), \beta \in \mu U\Sigma(\hat{s}, \hat{t})\}$$

is precisely $\mu U\Sigma(h(s, \hat{s}), h(t, \hat{t}))$. □

CLAIM 1. The set M is a correct set of unifiers for $\langle h(s, \hat{s}) = h(t, \hat{t})\rangle_T$.

PROOF. This follows immediately since by choice of α and β, α is a correct unifier for $\langle s = t\rangle_T$ and β is a correct unifier for $\langle \hat{s} = \hat{t}\rangle_T$, and α and β are variable disjoint. □

CLAIM 2. The set M is a minimal set of unifiers for $\langle h(s, \hat{s}) = h(t, \hat{t})\rangle_T$.

PROOF. Suppose otherwise so that there exist $\sigma_1, \sigma_2 \in M$ such that $\sigma_1 \leqslant_T \sigma_2[W]$, where $W = \mathrm{Var}(s, \hat{s}, t, \hat{t})$. Let $\sigma_1 \circ \beta_1$ and $\sigma_2 = \alpha_2 \circ \beta_2$ where α_1, α_2 are minimal in $\mu U\Sigma(s, t)$ and β_1, β_2 are minimal in $\mu U\Sigma(\hat{s}, \hat{t})$. Thus, $\alpha_1 \circ \beta_1 \leqslant_T \alpha_2 \circ \beta_2[W]$. By Lemma 2 below, this implies that $\alpha_1 \leqslant_T \alpha_2[W_1]$ and $\beta_1 \leqslant_T \beta_2[W_2]$ which contradicts the minimality of $\alpha_1, \alpha_2 \in \mu U\Sigma(s, t)$ and of $\beta_1, \beta_2 \in \mu U\Sigma(\hat{s}, \hat{t})$. □

CLAIM 3. The set M is a complete set of unifiers for $\langle h(s, \hat{s}) = h(t, \hat{t})\rangle_T$.

PROOF. Let δ be some unifier for $h(s, \hat{s})$ and $h(t, \hat{t})$. Since h is free we have $\delta s =_T \delta t$ and $\delta\hat{s} =_T \delta\hat{t}$. By definition, $\mu U\Sigma(s, t)$ is complete so there exists $\alpha \in \mu U\Sigma(s, t)$ such that $\delta \leqslant_T \alpha[W_1]$ where $W_1 = \mathrm{Var}(s, t)$. Similarly, there exists $\beta \in \mu U\Sigma(\hat{s}, \hat{t})$ such that $\delta \leqslant_T \beta[W_2]$ for $W_2 = \mathrm{Var}(\hat{s}, \hat{t})$. By construction α and β are completely variable disjoint so that by Lemma 3 below, $\delta \leqslant_T \alpha \circ \beta[W_1 \cup W_2]$ and $\alpha \circ \beta \in M$. □

Claims 1–3 and the fact that the cardinality of $\mu U\Sigma(h(s, \hat{s}), h(t, \hat{t}))$ is n^2 are exactly what is needed for the result. □

In the proof of Lemma 1, two technical lemmas were used. We turn to their proofs.

Two substitutions α and β are *completely variable disjoint* if

$$DOM(\alpha) \cap DOM(\beta) = \phi, \quad VCOD(\alpha) \cap VCOD(\beta) = \phi,$$
$$DOM(\alpha) \cap VCOD(\beta) = \phi \quad \text{and} \quad DOM(\beta) \cap VCOD(\alpha) = \phi,$$

where for any substitution σ, the *domain* of σ, $DOM(\sigma)$, is $\{x \in V | \sigma x \neq x\}$.

LEMMA 2. *Let* $\alpha_1 \circ \beta_1 \leqslant_T \alpha_2 \circ \beta_2[W]$ where α_1, α_2 and β_1, β_2 are completely variable disjoint. Then for

$$W_1 = DOM(\alpha_1) \cup DOM(\alpha_2) \quad \text{and} \quad W_2 = DOM(\beta_1) \cup DOM(\beta_2),$$
$$\alpha_1 \leqslant_T \alpha_2[W_1] \quad \text{and} \quad \beta_1 \leqslant_T \beta_2[W_2].$$

PROOF. From $\alpha_1 \circ \beta_1 \leqslant_T \alpha_2 \circ \beta_2[W]$ we have

$$\alpha_1 \circ \beta_1 =_T \lambda \circ \alpha_2 \circ \beta_2[W]$$

for some $\lambda \in \Sigma$. Then for all $x \in W_1$, $\alpha_1 \circ \beta_1 x =_T \alpha_1 x$ since α_1 and β_1 are variable disjoint, and for all $x \in W_1$,

$$\lambda \circ \alpha_2 \circ \beta_2 x =_T \lambda \circ \alpha_2 x$$

since α_2 and β_2 are variable disjoint. Hence, $\alpha_1 x =_T \lambda \circ \alpha_2 x$ for all $x \in W_1$ so that $\alpha_1 \leqslant_T \alpha_2[W_1]$. Similarly, since $\alpha_1 \circ \beta_1$ and $\alpha_2 \circ \beta_2$ are variable disjoint, we have for all $x \in W_2$,

$$\alpha_1 \circ \beta_1 x =_T \beta_1 \circ \alpha_1 x =_T \beta_1 x$$

and

$$\lambda \circ \alpha_2 \circ \beta_2 x =_T \lambda \circ \beta_2 \circ \alpha_2 x =_T \lambda \circ \beta_2 x.$$

Hence, $\beta_1 x =_T \lambda \circ \beta_2 x$ for all $x \in W_2$ so that $\beta_1 \leqslant_T \beta_2[W_2]$. □

LEMMA 3. *Let* α *and* β *be completely variable disjoint. If* $\delta \leqslant_T \alpha[W_1]$ *and* $\delta \leqslant_T \beta[W_2]$, *then* $\delta \leqslant_T \alpha \circ \beta[W_1 \cup W_2]$.

PROOF. Let $\delta =_T \lambda_1 \circ \alpha[W_1]$ and $\delta =_T \lambda_2 \circ \beta[W_2]$. Then for all $x \in W_1$,

$$\delta x =_T \lambda_1 \circ \alpha x =_T \lambda_1 \circ \alpha \circ \beta x,$$

since x is not in $DOM(\beta)$. Similarly, we have for all $x \in W_2$,

$$\delta x =_T \lambda_2 \circ \beta x =_T \lambda_2 \circ \beta \circ \alpha x$$

since x is not in $DOM(\alpha)$; in addition, for all $x \in W_2$,

$$\lambda_2 \circ \beta \circ \alpha x =_T \lambda_2 \circ \alpha \circ \beta x,$$

since $DOM(\alpha) \cap DOM(\beta) = \phi$. Hence, $\delta \leqslant_T \alpha \circ \beta[W_1]$ and $\delta \leqslant_T \alpha \circ \beta[W_2]$ so that $\delta \leqslant_T \alpha \circ \beta[W_1 \cup W_2]$. □

The main result now follows immediately from Lemma 1.

THEOREM. *If T is a suitable first-order equational theory that is not unitary, then T is not bounded.*

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